Ασκήσεις Στατιστικής (Εκτιμήτριες)

1	Let $X_1, \ldots, X_n$ be an *i.i.d.* sample with unknown distribution. Let further μ and $\sigma^2 < \infty$ be the expectation and variance of that distribution and $a_1, \ldots a_n \in \mathbb{R}$ constants satisfying $\sum_{i=1}^n a_i = 1$. (a) Prove that $\hat{\mu} = \sum_{i=1}^n a_i X_i$ is an unbiased estimator of μ. (b) Prove that for $n = 2$ the variance $\hat{\sigma}_\mu^2$ of $\hat{\mu}$ is minimized by $a_1 = a_2 = \frac{1}{2}$. (Ορολογία: expectation=μέση τιμή, variance=διασπορά, unbiased=αμερόληπτος)
2	The radius of a circle is measured by a method which generates a random error X with $X \sim \mathcal{N}(0, \sigma^2)$. Consider an *i.i.d.* sample of X with n elements. (a) What is the distribution of the radius R? (b) Prove that $\hat{\theta} = \frac{1}{n-1} \sum_{i=1}^n (R_i - \bar{R})^2$ is an unbiased estimator of the variance of R. (*Hint*: $\sum_{i=1}^n (R_i - \mu)^2 = \sum_{i=1}^n (R_i - \bar{R} + \bar{R} - \mu)^2$) (c) Specify an unbiased estimator of the area inside the circle.
3	Let $X_1, \ldots, X_5$ be an *i.i.d.* sample with unknown distribution. Let μ and $\sigma^2 < \infty$ be the expectation and variance of that distribution. Consider the following estimators of the expectation: (1) $\hat{\mu}_1 = X_1 - X_2 + X_3 - X_4 + X_5$. (2) $\hat{\mu}_2 = 0.3X_1 + 0.1X_2 + 0.3X_3 + 0.1X_4 + 0.3X_5$. (3) $\hat{\mu}_3 = 0.2 \sum_{i=1}^5 X_i$. (4) $\hat{\mu}_4 = X_3$. Check whether the estimators are unbiased. Order all unbiased estimators according to their efficiency.

4	Let X_1, X_2, X_3 denote a simple random sample for a population with mean μ and standard deviation σ. Define from this sample the following two estimators for the population mean: $$\hat{\mu}_1 = \frac{X_1 + 2X_2 + 2X_3}{5}, \quad \hat{\mu}_2 = \frac{X_1 + 3X_2 + X_3}{5}$$ You are asked to: $a)$ Verify that both estimators are unbiased. $b)$ Determine which one of the estimators is more efficient and compute its relative efficiency. $c)$ Find another estimator that is more efficient than any of them.
5	Suppose that $X \sim Binomial(n, \theta)$ for $0 < \theta < 1$. (a) Verify that the estimator $T(X) = X/n$ is unbiased for θ. (b) Consider $\tau(\theta) = 1/\theta$. Find an unbiased estimator of $\tau(\theta)$.
6	Suppose that $X_1, \ldots, X_n$ is a random sample from the distribution with pdf $$f_{X\|\theta}(x\|\theta) = \frac{3\theta^3}{(x+\theta)^4} \qquad 0 < x < \infty$$ and zero otherwise, for parameter $\theta > 0$. Find an unbiased estimator for θ, and the variance of this estimator.
7	**Bb. Exercise.** Prove that if $Y_1, Y_2, \ldots, Y_n$ are IID each with the U$[0, \theta]$ distribution, then $(n+1)M/n$ is unbiased for θ where $M := \max(Y_1, Y_2, \ldots, Y_n)$.
8	**Example 3** *Suppose that the reaction time to a certain stimulus has a uniform distribution on the interval from 0 to an unknown upper limit θ. Let $X_1, \ldots, X_n$ be the reaction times in a sample. Let* $$\hat{\Theta}_1 = \max(X_i), \quad \hat{\Theta}_2 = \frac{n+1}{n}\hat{\theta}_1 = \frac{n+1}{n}\max(X_i), \quad \hat{\Theta}_3 = 2\overline{X}.$$ *be three different estimators for paramenter θ. Discuss the unbiasedness or biasedness for each estimator.* *Compare the efficiency for estimators $\hat{\Theta}_1$, $\hat{\Theta}_2$ and $\hat{\Theta}_3$* *Show that $\hat{\Theta}_1$ is a consistent estimator* (Ορολογία: consistent=συνεπής)

9	**Example 6** *Let* $X_1, X_2, \ldots, X_n$ *be a sample from a population with mean* μ *and variance* σ^2*. Consider the two unbiased enstimators:* $$\hat{\mu}_1 = \overline{X} = \frac{X_1 + \cdots + X_n}{n}$$ $$\hat{\mu}_2 = \frac{X_1 + X_n}{2}$$ *Show that* $\hat{\mu}_1$ *is a consistent enstimator os* μ*, while* $\hat{\mu}_2$ *is not.*
10	**Example 10** *Suppose the errors in a measurement process are uniformly distributed between -1/2 and 1/2 grams. Compare the efficiency of the following two unbiased estimators:* $$\hat{\mu}_1 = \overline{X}$$ $$\hat{\mu}_2 = \frac{\min(X_i) + \max(X_i)}{2}$$